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Microbundles and bundles

by

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The following concerns a generalization of the Kister-Mazur representation theorem for microbundles, which says that any microbundle over a finite dimensional simplicial complex contains a (up to bundle isomorphism) unique bundle. More precisely, the purpose of this note is to prove

Theorem a) Let  $\mu : X \xrightarrow{s} E \xrightarrow{p} X$  be an  $\mathbb{R}^q$ -microbundle over a paracompact base space, and let  $U \subset X$  be a nbhd of some closed set  $A \subset X$ . Suppose  $\mu|_U$  is actually an  $\mathbb{R}^q$ -bundle. Then there is a nbhd  $E'$  of  $sX$  in  $E$  such that  $\xi : X \xrightarrow{s'} E' \xrightarrow{p'} X$  is an  $\mathbb{R}^q$ -bundle and  $\xi|_{U'} = \mu|_{U'}$  for some nbhd  $U'$  of  $A$ .

b) Suppose  $\xi_1, \xi_2$  are  $\mathbb{R}^q$ -bundles contained in the microbundle  $\mu$  and that  $\xi_1|_{U'} = \xi_2|_{U'}$  for some nbhd  $U'$  of  $A$ . Then there is a bundle isomorphism  $\xi_1 \approx \xi_2$  which is the identity over some nbhd  $U''$  of  $A$ .

A key point in the proof is the germ extension theorem below. This is stated for the trivial bundle over any base space  $X$  in Mazur [3] but is false unless some restrictions are placed on the base (or the germ). In the case where  $X$  is para-compact this theorem seems to follow from the general theory of dillation nbhds as developed in [3]. In any case a direct proof has been given below. It uses methods of Kister and Mazur generalized from the case where  $X$  is a simplex to the case where  $X$  is any topological space. Reputedly Mazur has used his theory of dillation nbhds to establish the representation theorem in the case where  $X$  is locally compact, normal and Lindelöf. Since any such space is of course paracompact his result is contained in ours.

The main results of this paper can be generalized to the case of numerable microbundles cf. [1]. A more general and detailed version will appear elsewhere.

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1. In the sequel we use the concepts and notations of Milnor [4] except for the following modifications. In stead of an isomorphism-germ  $\mu \Rightarrow \mu'$  of microbundles, we speak of an isogerm or, if  $\mu$  equals  $\mu'$ , of an autogerm. An embedding  $\mathcal{M} \rightarrow \mathcal{M}'$  of  $\mathbb{R}^q$ -bundles with base  $X$  is a bundle map (i.e. a map of total spaces preserving zero-sections and fibers) which is an open embedding. If it is onto, it is an automorphism.

We now consider autogerms and embeddings of the trivial  $\mathbb{R}^q$ -bundle over some paracompact (Hausdorff) space  $X$ :

(1) Lemma. Let  $\psi : X \times \mathbb{R}^q \Rightarrow X \times \mathbb{R}^q$  be an autogerm of the trivial  $\mathbb{R}^q$ -bundle over  $X$ . Then there is a bundle embedding  $\phi : X \times \mathbb{R}^q \rightarrow X \times \mathbb{R}^q$  whose germ is  $\psi$ .

For any real number  $t \geq 0$ , let  $D_t$  denote the closed ball of radius  $t$  centered at the origin in  $\mathbb{R}^q$ . Suppose  $\phi : X \times \mathbb{R}^q \rightarrow X \times \mathbb{R}^q$  is a bundle embedding,  $\phi(x, v) = (x, \phi_x(v))$ . Define the functions  $\rho_\phi, \sigma_\phi : X \times [0, \infty) \rightarrow \mathbb{R}$  by

$$\rho_\phi(x, t) = \inf \{ \|\phi_x(v)\| \mid v \in \dot{D}_t \}, \quad \sigma_\phi(x, t) = \sup \{ \|\phi_x(v)\| \mid v \in \dot{D}_t \}$$

Note that  $\rho_\phi$  and  $\sigma_\phi$  are non-negative functions which are positive on  $X \times (0, \infty)$ .

(2) Lemma. For any bundle embedding  $\phi : X \times \mathbb{R}^q \rightarrow X \times \mathbb{R}^q$  the functions  $\rho_\phi, \sigma_\phi$  are continuous.

The proofs of (1) and (2) are fairly straight forward (although not trivial) and will be omitted. Note that if  $\phi$  in (2) is only defined on a nbhd of  $X \times D_t$  (say), then of course (2) remains valid when  $\rho_\phi, \sigma_\phi$  are considered as functions on  $X \times [0, t]$ . With fixed second arguments we term  $\rho_\phi$  and  $\sigma_\phi$  the inner and the outer radius functions respectively. We also write  $E_t$  for  $X \times D_t$ . More generally, if  $f : X \rightarrow \mathbb{R}$  is any non-negative, continuous function, define  $E_f \subset X \times \mathbb{R}^q$  by

$$(x, v) \in E_f \Leftrightarrow (x, v) \in X \times \mathbb{R}^q \text{ and } \|v\| \leq f(x)$$

Then  $E_f$  is a closed nbhd of  $X \times 0$  in  $X \times \mathbb{R}^q$  and if  $f$  takes the constant value  $t$ , then  $E_f = E_t$ . The proof of (1) relies on the fact that  $X \times 0$  has a fundamental system of nbhds  $E_f$ .

(3) Lemma. Let  $\phi : X \times \mathbb{R}^q \rightarrow X \times \mathbb{R}^q$  be a bundle embedding. Then  
there is an embedding  $\phi' : X \times \mathbb{R}^q \rightarrow X \times \mathbb{R}^q$  with germ  $\phi' = \text{germ } \phi$   
such that  $\phi'E_1 \supset E_2$ .

Proof. There is the positive radius function  $\varrho = \varrho_\phi(\cdot, 1) : X \rightarrow \mathbb{R}$ .  
 Define  $F : X \times \mathbb{R}^q \rightarrow \mathbb{R}$  by

$$F(x, v) = \begin{cases} \frac{2}{\varrho(x)} \left( (\|v\| - \frac{1}{2}\varrho(x)) \frac{2}{\varrho(x)} + (\varrho(x) - \|v\|) \right), & \frac{1}{2}\varrho(x) \leq \|v\| \leq \varrho(x) \\ 1, & \|v\| \leq \frac{1}{2}\varrho(x) \\ \frac{2}{\varrho(x)}, & \varrho(x) \leq \|v\| \end{cases}$$

Then  $F$  is continuous and positive. Therefore the bundle map  
 $\phi' : X \times \mathbb{R}^q \rightarrow X \times \mathbb{R}^q$  given by

$$\phi'(x, v) = (x, F(x, v) \cdot \phi_x(v))$$

is an embedding, and clearly  $\text{germ } \phi' = \text{germ } \phi$ . Moreover  
 $\phi_x D_1 \supset D_{\varrho(x)}$  for all  $x$  in  $X$ , and multiplication of  
 $\phi_x(v) \in D_{\varrho(x)}$  by  $F(x, v)$  expands  $D_{\varrho(x)}$  to  $D_2$ . Thus  $\phi'E_1 \supset E_2$ .

(4) Theorem. Let  $\psi : X \times \mathbb{R}^q \Rightarrow X \times \mathbb{R}^q$  be an autogerm of the  
trivial  $\mathbb{R}^q$ -bundle over a paracompact space  $X$ . Then there is an  
automorphism  $\phi : X \times \mathbb{R}^q \rightarrow X \times \mathbb{R}^q$  whose germ is  $\psi$ .

Proof. By (1) and (3) we may assume the germ  $\psi$  represented by a  
 bundle embedding  $\phi_1 : X \times \mathbb{R}^q \rightarrow X \times \mathbb{R}^q$  such that  $\phi_1 E_1 \supset E_2$ .  
 We now proceed by an inductive argument. Suppose we have constructed  
 embeddings  $\phi_1, \phi_2, \dots, \phi_n : X \times \mathbb{R}^q \rightarrow X \times \mathbb{R}^q$ ,  $n \geq 1$ , such that

$$I_n \quad \phi_i E_i \supset E_{i+1} \quad i = 1, 2, \dots, n$$

$$II_n \quad \phi_i = \phi_{i-1} \text{ on } E_{i-1} \quad i = 2, 3, \dots, n$$

Since  $\phi_n^{-1}$  is defined on  $E_{n+1}$ , there is a positive inner  
 radius function  $1 = \varrho_{\phi_n^{-1}}(\cdot, n) : X \rightarrow \mathbb{R}$  measuring the inner radii  
 of  $\phi_n^{-1} E_n$ . Then  $\phi_n E_1 \subset E_n$ .

Similarly  $\phi_n$  has a positive outer radius function  $r = \phi_n(\cdot, n+1) : X \rightarrow \mathbb{R}$ . Then  $\phi_n E_{n+1} \subset E_r$ .

As in the proof of lemma (3) one now constructs positive, continuous functions  $G, H : X \times \mathbb{R}^q \rightarrow \mathbb{R}$  such that the bundle maps  $\Psi, \Theta : X \times \mathbb{R}^q \rightarrow X \times \mathbb{R}^q$  defined by

$$\Psi(x, v) = (x, G(x, v)v), \quad \Theta(x, v) = (x, H(x, v)v)$$

are automorphisms with the following properties:

$\Psi E_n = E_1$ ,  $\Psi = \text{identity outside } E_{n+1}$ ,  $\Psi E_{n+1} = E_{n+1}$ .

$\Theta E_{n+1} = E_r$ ,  $\Theta = \text{identity on } E_n$ .

Now, let  $\Pi : X \times \mathbb{R}^q \rightarrow X \times \mathbb{R}^q$  be defined by

$$\Pi = \begin{cases} \phi_n \Psi^{-1} \phi_n^{-1} & \text{on } \phi_n E_{n+1} \\ \text{identity} & \text{outside } \phi_n E_{n+1} \end{cases}$$

Then  $\Pi$  is a bundle automorphism with  $\Pi E_r = E_r$ . Finally form the composite embedding  $\phi_{n+1} = \Pi \Theta \phi_n \Psi$ . One checks that it satisfies the induction conditions  $I_{n+1}$ ,  $\Pi_{n+1}$ .

Thus there exists a sequence of bundle embeddings

$\phi_1, \phi_2, \dots : X \times \mathbb{R}^q \rightarrow X \times \mathbb{R}^q$  with germ  $\phi_1 = \varphi$ , and such that  $\phi_n$  satisfies  $I_n$  and  $\Pi_n$ ,  $n = 1, 2, \dots$ . Then there is a limit bundle map  $\phi = \varinjlim \phi_n$  which is obviously an embedding. By the requirements  $I_n$  it is onto and therefore actually an automorphism. By the requirements  $\Pi_n$   $\phi|_{E_1} = \phi_1|_{E_1}$  and therefore germ  $\phi = \text{germ } \phi_1 = \varphi$ .

2. For any bundle  $\mathcal{E} : X \xrightarrow{\pi} E \xrightarrow{p} X$  if  $A \subset X$ , then  $\mathcal{E}|_A : A \xrightarrow{\pi|_A} E|_A \xrightarrow{p|_A} A$  denotes the restriction of  $\mathcal{E}$  to  $A$ , and if  $\varphi$  is a bundle map germ from  $\mathcal{E}$  to  $\mathcal{E}'$ , then  $\varphi|(\mathcal{E}|_A)$  or  $\varphi|(E|_A)$  denotes the restriction of  $\varphi$  to  $\mathcal{E}|_A$ . The subset  $A \subset X$  is trivializing for  $\mathcal{E}$  in case  $\mathcal{E}|_A$  is isomorphic to the trivial  $\mathbb{R}^q$ -bundle over  $A$ . Such an isomorphism is called a trivialization of  $\mathcal{E}$  over  $A$  and written  $E|_A \cong \times \mathbb{R}^q$ . A partition of unity on  $(\pi_i, W_i)_{i \in J}$ ,  $W_i = \pi_i^{-1}(0, 1]$ , is trivializing for  $\mathcal{E}$  in case the open cover  $(W_i)_{i \in J}$  is trivializing for  $\mathcal{E}$ . Since  $X$  is paracompact any open cover of  $X$  admits subordinate trivializing partitions of

unity (cf. Dold [1]). Similar definitions apply in the case where  $\mathcal{E}$  is a microbundle.

(1) Lemma. Let  $\mathcal{E} : X \xrightarrow{S} E \xrightarrow{P} X$  be an  $\mathbb{R}^q$ -bundle and let  $\varphi : E \rightarrow X \times \mathbb{R}^q$  be an isogerm of  $\mathcal{E}$  to the trivial  $\mathbb{R}^q$ -bundle. Suppose there exists a trivializing partition of unity  $(\pi_i, W_i)_{i=1,2}$  for  $\mathcal{E}$  and a trivialization  $\phi_1 : E|_{W_1} \approx W_1 \times \mathbb{R}^q$  of  $\mathcal{E}$  over  $W_1$  whose germ is  $\varphi(E|_{W_1})$ . Then there exists a trivialization  $\phi : E \approx X \times \mathbb{R}^q$  of  $\mathcal{E}$  whose germ is  $\varphi$  and such that  $\phi(E|_{W_1 - W_2}) = \phi_1|(E|_{W_1 - W_2})$ .

Lemma (1) is the key step in all the inductive arguments which follow. It is a consequence mainly of the germ extension theorem. We omit the details in the proof and concentrate instead on a typical application:

(2) Corollary. Let  $\mathcal{E} : X \xrightarrow{S} E \xrightarrow{P} X$  be an  $\mathbb{R}^q$ -bundle and  $\varphi : E \rightarrow X \times \mathbb{R}^q$  an isogerm to the trivial  $\mathbb{R}^q$ -bundle. Then there exists a trivialization  $\phi : E \approx X \times \mathbb{R}^q$  of  $\mathcal{E}$  whose germ is  $\varphi$ .

Proof. Let  $(\pi_i, W_i)_{i \in J}$  be a trivializing partition of unity for  $\mathcal{E}$ . If  $K \subset J$ , write  $W_K = \bigcup_{i \in K} W_i$ ,  $\pi_K = \sum_{i \in K} \pi_i$ ,  $E_K = E|_{W_K}$ ,  $\varphi_K = \varphi|_{E_K}$ , etc. By paracompactness of  $X$  we may assume the  $\pi_i$  so "small" that even their closed supports  $\overline{W_i}$  are trivializing for  $\mathcal{E}$ . By (1.4) there is a trivialization of  $\mathcal{E}_i = \mathcal{E}|_{W_i}$

$$E_i \approx W_i \times \mathbb{R}^q$$

whose germ is  $\varphi_i$ . Consider the collection of pairs  $(K, \phi)$ , where  $K \subset J$  and  $\phi : E_K \approx W_K \times \mathbb{R}^q$  is a trivialization of  $\mathcal{E}_K$  with germ  $\varphi_K$ . Order this collection by defining  $(K, \phi) \leq (K', \phi')$  whenever the following is true:

- (a)  $K \subset K'$
- (b) If  $e \in E_K$  and  $\pi_K p(e) = \pi_{K'} p(e)$ , then  $\phi(e) = \phi'(e)$

One checks that the collection pairs so ordered

is inductive, and therefore each pair is contained in a maximal one. Let  $(K, \phi)$  be such a maximal pair. If  $K \neq J$ , let  $j \in J-K$  and write  $K' = K \cup \{j\}$ . Form the partition of unity  $(\pi_1, \pi_2)$  of  $W_{K'}$  with  $\pi_1 = \pi_K / (\pi_K + \pi_j)$ ,  $\pi_2 = \pi_j / (\pi_K + \pi_j)$ ; then  $W_1 = W_K$  and  $W_2 = W_j$ . Thus the bundle  $\xi_{K'}$  satisfies the conditions of lemma (2.1) with  $\phi_1$  equal  $\phi$  in  $(K, \phi)$  above and  $\psi$  equal  $\psi_{K'}$ . (The fact that  $W_{K'}$  is generally not paracompact is unimportant in this case.) It follows from (2.1) then that there exists a trivialization  $\phi'$  of  $\xi_{K'}$  whose germ is  $\psi_{K'}$  such that  $(K, \phi) \leq (K', \phi')$ . Clearly  $(K, \phi)$  is different from  $(K', \phi')$  contradicting the maximality of  $(K, \phi)$ . Thus we must have  $K = J$ , and so  $\phi: E_K = E \approx X \times \mathbb{R}^q$  is the required trivialization.

We are now ready for the proof of the main theorem announced in the introduction: Let  $\mu: X \xrightarrow{s} E \xrightarrow{p} X$  be a given  $\mathbb{R}^q$ -microbundle with trivializing partition of unity  $(\pi_i, W_i)_{i \in J}$ . Consider the collection of all triples  $(K, \xi, (\phi_i)_{i \in K})$  where  $K \subset J$ ,  $\xi: W_K \xrightarrow{s_K} E_K \xrightarrow{p_K} X$  is an  $\mathbb{R}^q$ -bundle contained in the microbundle  $\mu|_{W_K}$  (i.e.  $E_K$  is an open nbhd of  $s|_{W_K}$  in  $p^{-1}W_K$  ( $W_K = \bigcup_{i \in K} W_i$ )),  $p_K$  is the restriction of  $p$  to  $E_K$ , and  $s_K$  followed by the inclusion  $E_K \subset p^{-1}W_K$  equals  $s|_{W_K}$ , and  $(\phi_i)_{i \in K}$  is a family of trivializations  $\phi_i: E_K|_{W_i} \approx W_i \times \mathbb{R}^q$

$$\phi_i: E_K|_{W_i} \approx W_i \times \mathbb{R}^q$$

of  $W_i$ ,  $i \in K$ . In this non-empty collection introduce an order relation  $\leq$  by defining  $(K, \xi, (\phi_i)_{i \in K}) \leq (K', \xi', (\phi'_j)_{j \in K'})$  whenever the following is true:

- (a)  $K \subset K'$
- (b)  $x \in W_K$  &  $\pi_K(x) = \pi_{K'}(x) \Rightarrow p_K^{-1}x = p_{K'}^{-1}x$
- (c)  $e \in E_K$  &  $\pi_K p(e) = \pi_{K'} p(e) \Rightarrow \phi_i(e) = \phi'_i(e)$  for any  $i \in K$  such that  $p(e) \in W_i$ .

This order relation is in fact inductive, and so each triple is contained in a maximal triple by Zorn's lemma. Let  $(K, \xi, (\phi_i)_{i \in K})$  be a maximal triple. We aim to show that  $K = J$ , which will prove that any microbundle contains a bundle.

Suppose that  $K \neq J$  and let  $j \in J-K$ . Since  $W_j$  is trivializing for the microbundle  $\mu$ , there is a trivial bundle  $\eta: W_j \xrightarrow{s_j} E_j \xrightarrow{p_j} W_j$  contained in  $\mu|_{W_j}$ . Let  $\psi: E_j \approx W_j \times \mathbb{R}^q$  be

a trivialization of  $\eta$ . Then  $(j, \eta, \Psi)$  is a triple. We wish to enlarge  $(K, \xi, (\phi_i))$  by gluing  $(j, \eta, \Psi)$  to it, thereby reaching a contradiction. To do this we need a bundle isomorphism between  $\xi|_{(W_K \cap W_j)}$  and  $\eta|_{(W_K \cap W_j)}$ . Since these two bundles determine the same microbundle (up to equivalence) and this microbundle is trivial, by (2.2) there is an isomorphism  $\Gamma : E_K|_{(W_K \cap W_j)} \approx E_j|_{(W_K \cap W_j)}$  whose germ is the identity. (It is not important that  $W_K \cap W_j$  may not be paracompact.) Gluing  $\xi$  and  $\eta$  together along  $\Gamma$  gives a bundle  $\xi \vee_{\Gamma} \eta : W_{K'} \xrightarrow{s_{K'}} E_K \vee_{\Gamma} E_j \xrightarrow{p_{K'}} W_{K'}$ , with  $K' = K \cup \{j\}$ , such that  $(\xi \vee_{\Gamma} \eta)|_{(W_K - W_j)} = \xi_K|_{(W_K - W_j)}$ . Unfortunately  $\xi \vee_{\Gamma} \eta$  is not contained in  $\mu|_{W_{K'}}$ . There is however an open nbhd  $U$  of the zero-section  $s_{K'}|_{W_{K'}}$  in  $E_K \vee_{\Gamma} E_j$  with  $U|_{(W_K - W_j)} = E_K|_{(W_K - W_j)}$  which is contained in  $E|_{W_{K'}}$ . By the paracompactness of  $X$  we may assume  $U$  such that there exists a positive continuous function  $f : W_{K'} \rightarrow \mathbb{R}_{\infty}$  ( $\mathbb{R}_{\infty}$  = extended real line) with  $f|_{(W_K - W_j)}$  constant equal to  $\infty$  such that if  $E_f|_{W_j} \subset E_j$  denotes the nbhd of  $s_j|_{W_j}$  corresponding under  $\Psi : E_j \approx W_j \times \mathbb{R}^q$  to the  $f|_{W_j}$ -nbhd of  $W_j \times 0$  in  $W_j \times \mathbb{R}^q$ , then  $E_f|_{W_j}$  is contained in  $U|_{W_j}$ . Form

$$E_{K'} = E_K|_{(W_K - W_j)} \cup \text{interior of } E_f|_{W_j} \text{ in } E_j$$

Then  $E_{K'} \subset E|_{W_{K'}}$ , and there is an  $\mathbb{R}^q$ -bundle  $\xi' : W_{K'} \xrightarrow{s_{K'}} E_{K'} \xrightarrow{p_{K'}} W_{K'}$  contained in  $\mu|_{W_{K'}}$ . In fact trivializations  $\phi'_i, \psi', i \in K$ , for  $\xi'$  over  $W_i, W_j, i \in K$ , can be constructed such that  $\phi'_i|_{(W_i - W_j)} = \phi_i|_{(W_i - W_j)}$ . It follows that  $(K', \xi', (\phi'_i, \psi'))$  is a triple majorizing  $(K, \xi, (\phi_i))$ . The contradiction shows that for any maximal triple  $(K, \xi, (\phi_i))$   $K = J$ . Thus  $\xi$  is a bundle over  $W_J = X$  contained in  $\mu$ . Finally notice that if  $\mu, U$  and  $A$  are as described under a) in the representation theorem, then there exists a trivializing partition of unity  $(\pi_i|_{W_i})_{i \in J}$  for  $\mu$ , a nbhd  $U'$  of  $A$  in  $U$  and a  $K \subset J$  with  $W_K \subset U'$  such that if  $i \in J - K$ , then  $W_i \cap U' = \emptyset$ . Thus there is a maximal triple majorizing  $(K, \mu|_{W_K}, (\phi_i))$ , say  $(J, \xi, (\psi_i))$ , and by definition of the ordering  $\xi|_{U'} = \mu|_{U'}$ . This proves part a) in the theorem.

Part b) follows similarly from an inductive argument. This ends the proof of the representation theorem for microbundles.



3. We consider some consequences of the representation theorem.

(1) Theorem. Let  $\xi: X \xrightarrow{s} E \xrightarrow{p} X$  be an  $\mathbb{R}^q$ -bundle over a paracompact base space  $X$ . Then there is a fibre homotopy  $H: \text{id}_E \simeq \text{sp rel } sX$  such that for  $t \neq 1$   $H|_{E \times \{t\}} \simeq E$  is a bundle embedding.

This does not follow from the representation theorem but rather by an inductive argument as shown above.

(2) Corollary. Let  $\mu: X \xrightarrow{s} E \xrightarrow{p} X$ ,  $\nu: E \xrightarrow{t} E' \xrightarrow{q} E$  be microbundles over  $X$  paracompact. Then the composite microbundle  $\mu \circ \nu$  is isomorphic to  $\mu \oplus s^* \nu$ . Similarly, for microbundles  $\mu, \mu'$  over  $X$ ,  $\mu \oplus \mu'$  is isomorphic to  $\mu \circ p^* \mu'$ .

Proof. For the first part of the corollary a proof of Milnor [5] for PL-microbundles carries over once we can find a nbhd  $U$  of  $sX$  in  $E$  such that  $s \circ (p|_U)$  is homotopic to the inclusion  $U \subset E$ , and this certainly follows from the representation theorem together with (3.1). The last part follows from the first.

If  $\mu$  and  $\nu$  are actually bundles the composite, although a microbundle, need not be a bundle. By the representation theorem it does contain an essentially unique bundle, however, which could be called the composite bundle, and which is bundle isomorphic to the Whitney sum of bundles  $\mu \oplus s^* \nu$ . This is still true if the word "bundle" is replaced by "orthogonal bundle".

If  $\xi$  is an  $\mathbb{R}^q$ -bundle with a trivializing partition of unity then there is an  $S^q$ -bundle  $\xi_\infty$ , determined up to natural isomorphism, with two sections  $s_0, s_\infty$ . The  $\mathbb{R}^q$ -bundle  $\xi$  is contained in  $\xi_\infty$  in such a way that the zero section corresponds to  $s_0$  and the total space  $E$  to  $E_\infty - \text{ims}_\infty$ . If  $X$  is the base of  $\xi$  (and  $\xi_\infty$ ) and  $A \subset X$  is any subspace, define the Thom space  $T_\xi(X, A)$  to be the pointed space

$$T_\xi(X, A) = E_\infty / (\text{ims}_\infty \cup p_\infty^{-1} A),$$

the collapsed subset  $\text{ims}_\infty \cup p_\infty^{-1} A$  serving as base point. By the representation theorem any microbundle over a paracompact base gets Thom spaces unique up to homeomorphism. In another note we use this concept to extend the Atiyah-Bott-Shapiro S-duality theorem to microbundles over topological manifolds.

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